

# ON THE STABILITY OF THE WULFF SHAPE

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**ABSTRACT.** Given a positive function  $F$  on  $\mathbb{S}^n$  satisfying an appropriate convexity assumption, we consider hypersurfaces for which a linear combination of some higher order anisotropic curvatures is constant. We define the variational problem for which these hypersurfaces are critical points and we prove that, up to translations and homotheties, the Wulff shape is the only stable closed hypersurface of the Euclidean space for this problem.

## 1. INTRODUCTION

The stability of hypersurfaces for volume preserving variational problem has a long history since the first result for the stability of constant mean curvature in the Euclidean space by Barbosa and do Carmo [3]. Many authors have been interested in stability problems in various contexts, like for other space forms and/or higher order mean curvatures (see [1, 2, 4, 5, 7, 8, 9, 10, 12] for instance). In 2013, Velázquez, de Sousa and de Lima [21] defined the notion of  $(r, s)$ -stability which generalizes the classical notion of stability for the mean curvature or  $r$ -stability for higher order mean curvatures. They prove that the only compact  $(r, s)$ -stable hypersurfaces in the sphere or the hyperbolic space are geodesic hyperspheres. This result was recently extended for hypersurfaces of the Euclidean space by da Silva, de Lima and Velázquez [20].

On the other hand, during the last decade, an intensive interest has been brought to the study of hypersurfaces of Euclidean spaces in an anisotropic setting. Many of the classical characterizations of the geodesic hyperspheres have an analogue with the Wulff Shape as characteristic hypersurface, like anisotropic Hopf or Alexandrov-type theorems (see [11, 13, 14, 16, 17, 18]). In particular, in [17], Onat proved that a closed convex hypersurface of the Euclidean space with linearly related anisotropic mean curvatures  $H_r^F = a_0 H_0^F + a_1 H_1^F + a_2 H_2^F + \cdots + a_{r-1} H_{r-1}^F$  is the Wulff shape. The hypothesis that  $X(M)$  is convex is crucial in Onat's result.

The aim of this short note is to prove an anisotropic analogue to [20]. This extend to the anisotropic  $(r, s)$ -stability the results of [18], [13] and [6] and give an other characterization of the Wulff shape as the only hypersurface (up to translations and homotheties) which have linearly related anisotropic mean curvatures, without assuming that  $X(M)$  is convex. Namely, we prove the following

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**Theorem 1.1.** *Let  $n, r, s$  three integers satisfying  $0 \leq r \leq s \leq n - 2$ . Let  $F : \mathbb{S}^n \rightarrow \mathbb{R}_+$  be a smooth function satisfying the following convexity assumption*

$$A_F = (\nabla dF + F \text{Id}_{|T_x \mathbb{S}^n})_x > 0,$$

*for all  $x \in \mathbb{S}^n$  and let  $X : M^n \rightarrow \mathbb{R}^{n+1}$  be a closed hypersurface with positive anisotropic  $(s+1)$ -th mean curvature  $H_{s+1}^F$ . Assume that the quantity  $\sum_{j=r}^s a_j b_j H_{j+1}^F$  is constant, where  $a_r, \dots, a_s$  are some nonnegative constants (with at least one non zero) and  $b_j = (j+1) \binom{n}{j+1}$  for any  $j \in \{r, \dots, s\}$ .*

*Then  $X : M^n \rightarrow \mathbb{R}^{n+1}$  is  $(r, s, F)$ -stable if and only if  $X(M)$  is the Wulff shape  $W_F$ , up to translations and homotheties.*

The notion of  $(r, s, F)$ -stability will be defined in Section 2.2.

## 2. PRELIMINARIES

**2.1. Anisotropic mean curvatures.** Here, we recall the basics of anisotropic mean curvatures. These facts are classical, hence, we will not recall their proofs. First, let  $F : \mathbb{S}^n \rightarrow \mathbb{R}_+$  be a smooth function satisfying the following convexity assumption

$$(1) \quad A_F = (\nabla dF + F \text{Id}_{|T_x \mathbb{S}^n})_x > 0,$$

Let  $(M^n, g)$  be a compact, connected, oriented Riemannian manifold without boundary isometrically immersed into  $\mathbb{R}^{n+1}$  by  $X$  and denote by  $\nu$  a normal unit vector field. Let  $X^T = X - \langle X, \nu \rangle \nu$  be its projection on the tangent bundle of  $X(M)$ .

The (real-valued) second fundamental form  $B$  of the immersion is defined by

$$B(X, Y) = \langle \bar{\nabla}_X \nu, Y \rangle,$$

where  $\langle \cdot, \cdot \rangle$  and  $\bar{\nabla}$  are respectively the Riemannian metric and the Riemannian connection of  $\mathbb{R}^{n+1}$ . We also denote by  $S$  the Weingarten operator, which is the  $(1, 1)$ -tensor associated with  $B$ .

Let  $S_F = A_F \circ S$ . The operator  $S_F$  is called the  $F$ -Weingarten operator and its eigenvalues  $\kappa_1, \dots, \kappa_n$  are the anisotropic principal curvatures. Now let us recall the definition of the anisotropic high order mean curvature  $H_r^F$ . First, we consider an orthonormal frame  $\{e_1, \dots, e_n\}$  of  $T_x M$ . For all  $k \in \{1, \dots, n\}$ , we set

$$\sigma_r = \binom{n}{r}^{-1} \sum_{\substack{1 \leq i_1, \dots, i_r \leq n \\ 1 \leq j_1, \dots, j_r \leq n}} \epsilon \left( \begin{matrix} i_1 \cdots i_r \\ j_1 \cdots j_r \end{matrix} \right) S_{i_1 j_1}^F \cdots S_{i_r j_r}^F,$$

where the  $S_{ij}^F$  are the coefficients of the  $F$ -Weingarten operator. The symbols  $\epsilon \left( \begin{matrix} i_1 \cdots i_r \\ j_1 \cdots j_r \end{matrix} \right)$  are the usual permutation symbols which are zero if the sets  $\{i_1, \dots, i_r\}$  and  $\{j_1, \dots, j_r\}$  are different or if there exist distinct  $p$  and  $q$  with  $i_p = i_q$ . For all other cases,  $\epsilon \left( \begin{matrix} i_1 \cdots i_r \\ j_1 \cdots j_r \end{matrix} \right)$  is the signature of the permutation

$\begin{pmatrix} i_1 \cdots i_r \\ j_1 \cdots j_r \end{pmatrix}$ . Then, the  $r$ -th anisotropic mean curvature of the immersion is defined by

$$H_r^F = \begin{pmatrix} n \\ r \end{pmatrix}^{-1} \sigma_r.$$

We denote simply  $H^F$  the anisotropic mean curvature  $H_1^F$ . Moreover, for convenience, we set  $H_0^F = 1$  et  $H_{n+1}^F = 0$  by convention.

For  $r \in \{1, \dots, n\}$ , the symmetric  $(1, 1)$ -tensor associated to  $H_r^F$  is

$$P_r = \frac{1}{r!} \sum_{\substack{1 \leq i, i_1, \dots, i_r \leq n \\ 1 \leq j, j_1, \dots, j_r \leq n}} \epsilon \begin{pmatrix} i, i_1 \cdots i_r \\ j, j_1 \cdots j_r \end{pmatrix} S_{i_1 j_1}^F \cdots S_{i_r j_r}^F e_i^* \otimes e_j^*.$$

We also define the following useful operators  $T_r = P_r A_F$ . Note that  $T_r$  is symmetric. Moreover, we have these classical facts about the anisotropic mean curvatures (see [10] for instance).

**Lemma 2.1.** *For any  $r$ , we have*

- (1)  $P_r$  is divergence-free,
- (2)  $\text{tr}(P_r) = (n - r)\sigma_r$ ,
- (3)  $\text{tr}(P_r S_F) = (r + 1)\sigma_{r+1}$ ,
- (4)  $\text{tr}(P_r S_F^2) = \sigma_1 \sigma_{r+1} - \sigma_{r+2}$ .

**Lemma 2.2.** *Let  $r \in \{1, \dots, n - 1\}$ . If  $H_{r+1} > 0$ , then For all  $j \in \{1, \dots, r\}$ ,*

- (1)  $H_j^F > 0$ ,
- (2)  $H^F H_{j+1}^F - H_{j+2}^F \geq 0$ . *Moreover, equality occurs at a point  $p$  if and only if all the anisotropic principal curvatures at  $p$  are equal. Hence, equality occurs everywhere if and only if  $M$  is the Wulff shape, up to translations and homotheties.*

Finally, we recall the anisotropic analogue of the classical Hsiung-Minkowski formulas [15]. The proof can be found in [10] and uses in particular the fact that  $P_r$  is divergence-free.

**Lemma 2.3.** *Let  $r \in \{0, \dots, n - 1\}$ . Then, we have*

$$\int_M (F(\nu) H_r^F + H_{r+1}^F \langle X, \nu \rangle) dv_g.$$

**2.2. The variational problem.** In this section, we describe the stability problem that we will consider. For this we introduce the anisotropic  $r$ -area functionals

$$(2) \quad \mathcal{A}_{r,F} = \left( \int_M F(\nu) \sigma_r dv_g \right),$$

for  $r \in \{0, \dots, n - 1\}$  and where  $dv_g$  denotes the Riemannian volume form on  $M$ . Now we consider a variation of the immersion  $X$ . Precisely, let  $\varepsilon > 0$  and

$$(3) \quad \mathcal{X}: (-\varepsilon, \varepsilon) \times M \longrightarrow \mathbb{R}^{n+1},$$

such that for all  $t \in (-\varepsilon, \varepsilon)$ ,  $\mathcal{X}_t := \mathcal{X}(t, \cdot)$  is an immersion of  $M$  into  $\mathbb{R}^{n+1}$  and  $\mathcal{X}(0, \cdot) = X$ . We denote by  $\sigma_r(t)$  the corresponding curvature functions, by  $\mathcal{A}_{r,F}(t)$  the  $r$ -area of  $\mathcal{X}_t$  and finally we set

$$(4) \quad f_t = \left\langle \frac{dX}{dt}, \nu_t \right\rangle,$$

where  $\nu_t$  is the unit normal to  $M$  induced by  $\mathcal{X}_t$ . Finally, we denote by  $g_t$  the induced metric on  $M$ .

Note that

$$(5) \quad \mathcal{A}'_{r,F}(t) = -b_{r+1} \int_M f H_{r+1}^F(t) dv_{g_t},$$

where  $b_{r+1} = (r+1) \binom{n}{r+1}$  cf. [13]. We also consider the volume functional

$$(6) \quad V(t) = \int_{[0,t] \times M} \mathcal{X}^* dv.$$

It is easy to see, cf. [4, Lemma 2.1], that  $V$  satisfies

$$(7) \quad V'(t) = \int_M f_t dv_{g_t}$$

and so  $\mathcal{X}$  preserves the volume if and only if  $\int_M f_t dv_{g_t} = 0$  for all  $t$ . Moreover, according to [4, Lemma 2.2], for any function  $f_0 : M \rightarrow \mathbb{R}$  such that  $\int_M f_0 dv_g = 0$ , there exists a variation of  $X$  preserving the volume and with normal part given by  $f_0$ .

Now, let  $r$  and  $s$  two integers satisfying  $0 \leq r \leq s \leq n-2$  and  $a_j, j = r, \dots, s$  some nonnegative real numbers with at least one non zero. We consider the following anisotropic  $(r, s)$ -area functional  $\mathcal{B}_{r,s,F}$  defined by

$$(8) \quad \mathcal{B}_{r,s,F} = \sum_{j=r}^s a_j \mathcal{A}_{j,F}.$$

This functional appears naturally when considering hypersurfaces with linearly related higher order anisotropic mean curvatures. Indeed, we consider variations of  $M$  that preserve the balanced volume, the Jacobi functional associated with this anisotropic  $(r, s)$ -area is given by

$$\begin{aligned} \mathcal{J}_{r,sF} : (\varepsilon, \varepsilon) &\longrightarrow \mathbb{R} \\ t &\longmapsto \mathcal{B}_{r,s,F}(t) + \Lambda V(t), \end{aligned}$$

where  $\Lambda$  is a constant to be determined. From (5) and (7), we have immediately

$$\mathcal{J}'_{r,sF}(t) = \int_M \left( - \sum_{j=r}^s a_j b_j H_{j+1}^F + \Lambda \right) f_t dv_{g_t}.$$

Hence, we have, like in the isotropic context, that  $X$  satisfies  $\sum_{j=r}^s a_j b_j H_{j+1}^F = \text{constant}$  if and only if  $X$  is a critical point of the functional  $\mathcal{J}_{r,s,F}$ , or equivalently, if and only if  $X$  is a critical point of  $\mathcal{B}_{r,s,F}$  for variations that preserve the balanced volume. Now, we give the definition of the anisotropic  $(r, s)$ -stability that we call  $(r, s, F)$ -stability.

**Definition 2.4.** Let  $n, r, s$  three integers satisfying  $0 \leq r \leq s \leq n - 2$ . Let  $F : \mathbb{S}^n \rightarrow \mathbb{R}_+$  be a smooth function satisfying the following convexity assumption (1) and let  $X : M^n \rightarrow \mathbb{R}^{n+1}$  be a closed hypersurface satisfying

$$\sum_{j=r}^s a_j b_j H_{j+1}^F = \text{constant}.$$

Then,  $X$  is said  $(r, s, F)$ -stable if  $\mathcal{B}_{r,s,F}''(0) \geq 0$  for all volume-preserving variations of  $X$ .

We consider the Jacobi operator  $\mathcal{J}_{r,s,F}''(0)$  defined on the set  $\mathcal{F}$  of smooth functions on  $M$  with  $\int_M f dv_g = 0$ . From the definitions, we have clearly  $\mathcal{B}_{r,s,F}''(0) = \mathcal{J}_{r,s,F}''(0)[f]$  where  $f \in \mathcal{F}$  defines the variation  $\mathcal{X}$ . Therefore, the  $(r, s, F)$ -stability corresponds to the nonnegativity of the Jacobi operator.

We finish this section by giving the second variation formula for this variational problem.

**Lemma 2.5.** For any variation of mathematical  $X$  of  $X$  preserving the balanced volume, the second variation formula of  $\mathcal{B}_{r,s,F}$  at  $t = 0$  is given by

$$\mathcal{B}_{r,s,F}''(0) = \mathcal{J}_{r,s,F}''(0)[f] = - \sum_{j=r}^s (j+1) a_j \int_M \left( L_j f + \langle T_j \circ d\nu, d\nu \rangle f \right) f dv_g,$$

where  $f \in \mathcal{F}$  is the normal part of the variation  $\mathcal{X}$ .

*Proof:* The proof comes directly from the second variation formula for each functional  $\mathcal{A}_{j,F}$ . Indeed, from [13], we have

$$\mathcal{A}_{j,F}''(0) = -(j+1) a_j \int_M \left( L_j f + \langle T_j \circ d\nu, d\nu \rangle f \right) f dv_g$$

Then, we have just to multiply by  $a_j$  and sum from  $r$  to  $s$  to get the result.  $\square$

**2.3. Some lemmas.** Now, we fix some notations and give some useful lemmas. First, we define the following operators. For any  $f \in \mathcal{C}^\infty$ , we set

$$I_{j,F}[f] = L_j f + \langle T_j \circ d\nu, d\nu \rangle f$$

and

$$\mathcal{R}_{r,s,F} = \sum_{j=r}^s (j+1) a_j I_{j,F}[f].$$

Obviously from this definition and Lemma 2.5, we have

$$\mathcal{J}_{r,s,F}''(0)[f] = - \int_M f \mathcal{R}_{r,s,F}[f] dv_g.$$

First, we recall this lemma due to He and Li [13]. The proof of this lemma follows the idea of [19] and uses Lemma 2.1.

**Lemma 2.6.** For any  $j \in \{r, \dots, s\}$ , we have

- (1)  $I_{j,F}[\langle X, \nu \rangle] = -\langle \text{grad } \sigma_{j+1}, X^T \rangle - (j+1) \sigma_{j+1},$
- (2)  $I_{j,F}[F(\nu)] = -\langle \text{grad } \sigma_{j+1}, (\text{grad}_{\mathbb{S}^n} F) \circ \nu \rangle + \sigma_1 \sigma_{j+1} - (j+2) \sigma_{j+2}.$

Now, we have this last lemma about the symmetry of  $\mathcal{R}_{r,s,F}$  w.r.t. the  $L_2$ -scalar product. Namely, we have

**Lemma 2.7.** *For any two smooth functions  $f$  and  $h$  over  $M$ , we have*

$$\int_M h \mathcal{R}_{r,s,F}[f] dv_g = \int_M f \mathcal{R}_{r,s,F}[h] dv_g.$$

*Proof:* The proof is fairly standard. First, we compute

$$\begin{aligned} \int_M h L_j f dv_g &= \int_M h \operatorname{div} (T_j \nabla f) dv_g \\ &= - \int_M \langle T_j \nabla f, \nabla h \rangle dv_g \\ &= - \int_M \langle T_j \nabla h, \nabla f \rangle dv_g \\ &= \int_M f \operatorname{div} (T_j \nabla h) dv_g \\ &= \int_M f L_j h dv_g, \end{aligned}$$

where we have used the symmetry of  $T_j$  and the divergence theorem.

Hence, from the definition of  $I_{j,F}$  and the above identity, we get immediately that

$$\int_M h I_{j,F}[f] dv_g = \int_M f I_{j,F}[h] dv_g.$$

Finally, multiplying by  $a_j b_j$  and aking the sum over  $j$  from  $r$  to  $s$ , we get

$$\int_M h \mathcal{R}_{r,s,F}[f] dv_g = \int_M f \mathcal{R}_{r,s,F}[h] dv_g.$$

This concludes the proof.  $\square$

Now, we have all the ingredients to prove the main result of this note.

### 3. PROOF OF THEOREM 1.1

First, it is not difficult to see that the Wulff shape  $W_F$  is  $(r, s, F)$ -stable. Indeed, the Wulff shape has all its mean curvatures  $H_j^F$  constant and is  $(j, F)$ -stable, that

is,  $\mathcal{A}_{j,F}''(0) \geq 0$ , for any  $j \in \{0, \dots, n-1\}$  (see [13]). Therefore,  $\sum_{j=r}^s a_j b_j H_{j+1}^F$  is clearly constant and since the constants  $a_j$  are nonnegative, it is also clear that

$$\mathcal{B}_{r,s,F}''(0) = \sum_{j=r}^s a_j \mathcal{A}_{j,F}''(0) \geq 0,$$

which means that the Wulff shape is  $(r, s, F)$ -stable.

Conversely, suppose that  $X : M^n \rightarrow \mathbb{R}^{n+1}$  is  $(r, s, F)$ -stable. By definition, we have  $\mathcal{J}_{r,s,F}''(0)[f] \geq 0$  for any smooth function on  $M$  satisfying  $\int_M f dv_g = 0$ . We choose the particular test function  $f$  defined by

$$f = \alpha F(\nu) + \beta \langle X, \nu \rangle,$$

with  $\alpha = \frac{\int_M \left( \sum_{j=r}^s a_j b_j F(\nu) H_j^F \right) dv_g}{\int_M F(\nu)}$  and  $\beta = \sum_{j=r}^s a_j b_j H_{j+1}^F$ . First, remark that  $\beta$  is a constant by assumption. Moreover, using the anisotropic Hsiung-Minkowski formulas (Lemma 2.3) we have

$$\begin{aligned}
\int_M f dv_g &= \int_M (\alpha F(\nu) + \beta \langle X, \nu \rangle) dv_g \\
&= \alpha \int_M F(\nu) dv_g + \int_M \left( \sum_{j=r}^s a_j b_j H_{j+1}^F \langle X, \nu \rangle \right) dv_g \\
&= \alpha \int_M F(\nu) dv_g + \sum_{j=r}^s a_j b_j \int_M H_{j+1}^F \langle X, \nu \rangle dv_g \\
&= \alpha \int_M F(\nu) dv_g - \sum_{j=r}^s a_j b_j \int_M F(\nu) H_j^F dv_g \\
&= \alpha \int_M F(\nu) dv_g - \int_M \left( \sum_{j=r}^s a_j b_j F(\nu) H_j^F \right) dv_g \\
&= \alpha \int_M F(\nu) dv_g - \alpha \int_M F(\nu) dv_g = 0
\end{aligned}$$

Hence, the integral of  $f$  vanishes and  $f$  is eligible as a test function. Hence, we have  $\mathcal{J}_{r,s,F}''(0)[f] = - \int_M f \mathcal{R}_{r,s,F}[f] dv_g \geq 0$ . Now, let's compute  $\mathcal{R}_{r,s,F}[f]$ . We have

$$\begin{aligned}
\mathcal{R}_{r,s,F}[f] &= \sum_{j=r}^s (j+1) a_j I_{j,F}[f] \\
&= \sum_{j=r}^s (j+1) a_j I_{j,F}[\alpha F(\nu) + \beta \langle X, \nu \rangle] \\
&= \sum_{j=r}^s (j+1) a_j \left( \alpha I_{j,F}[F(\nu)] + \beta I_{j,F}[\langle X, \nu \rangle] \right).
\end{aligned}$$

From Lemma 2.6, we have

$$\begin{aligned}
\mathcal{R}_{r,s,F}[f] &= \sum_{j=r}^s (j+1) a_j \left[ \alpha \left( - \langle \text{grad } \sigma_{j+1}, (\text{grad}_{\mathbb{S}^n} F) \circ \nu \rangle + \sigma_1 \sigma_{j+1} - (j+2) \sigma_{j+2} \right) \right. \\
&\quad \left. + \beta \left( - \langle \text{grad } \sigma_{j+1}, X^T \rangle - (j+1) \sigma_{j+1} \right) \right]
\end{aligned}$$

Since  $\sum_{j=r}^s (j+1)a_j\sigma_{j+1} = \sum_{j=r}^s a_j b_j H_{j+1}^F = \text{constant}$ , we get

$$\begin{aligned}\mathcal{R}_{r,s,F}[f] &= \sum_{j=r}^s (j+1)a_j \left[ \alpha \left( \sigma_1 \sigma_{j+1} - (j+2)\sigma_{j+2} \right) - \beta(j+1)\sigma_{j+1} \right] \\ &= \sum_{j=r}^s a_j b_j \left[ \alpha \left( n H^F H_{j+1}^F - (n-j-1)H_{j+2}^F \right) - \beta(j+1)H_{j+1}^F \right]\end{aligned}$$

Moreover, we have

$$\begin{aligned}\mathcal{J}_{r,s,F}''(0)[f] &= - \int_M f \mathcal{R}_{r,s,F}[f] dv_g \\ &= - \int_M \left( \alpha F(\nu) + \beta \langle X, \nu \rangle \right) \mathcal{R}_{r,s,F}[f] dv_g \\ (9) \quad &= - \int_M \left( \alpha F(\nu) \mathcal{R}_{r,s,F}[f] + \beta f \mathcal{R}_{r,s,F}[\langle X, \nu \rangle] \right) dv_g.\end{aligned}$$

where we have used Lemma 2.7. The first term in this integral is

$$\begin{aligned}\alpha F(\nu) \mathcal{R}_{r,s,F}[f] &= \sum_{j=r}^s a_j b_j F(\nu) \left[ \alpha^2 \left( n H^F H_{j+1}^F - (n-j-1)H_{j+2}^F \right) - \alpha \beta(j+1)H_{j+1}^F \right] \\ &= \sum_{j=r}^s a_j b_j F(\nu) \left[ \alpha^2 (n-j-1)(H^F H_{j+1}^F - H_{j+2}^F) + \alpha^2 (j+1)H^F H_{j+1}^F \right. \\ (10) \quad &\quad \left. - \alpha \beta(j+1)H_{j+1}^F \right].\end{aligned}$$

The second term in (9) is, using Lemma 2.6 and the fact that  $\sum_{j=r}^s a_j b_j H_{j+1}^F$  is constant,

$$\begin{aligned}\beta f \mathcal{R}_{r,s,F}[\langle X, \nu \rangle] &= \beta f \sum_{j=r}^s (j+1)a_j I_{j,F}[\langle X, \nu \rangle] \\ &= \beta f \sum_{j=r}^s (j+1)a_j \left( - \langle \text{grad } \sigma_{j+1}, X^T \rangle - (j+1)\sigma_{j+1} \right) \\ &= -\beta f \sum_{j=r}^s a_j (j+1)^2 \sigma_{j+1} \\ &= -\beta f \sum_{j=r}^s a_j b_j (j+1)H_{j+1}^F \\ (11) \quad &= - \sum_{j=r}^s a_j b_j (j+1) \beta \left( \alpha F(\nu) + \beta \langle X, \nu \rangle \right) H_{j+1}^F.\end{aligned}$$



Now, putting (9) and (10) in (11), we get

$$\begin{aligned}
\mathcal{J}_{r,s,F}''(0)[f] &= - \sum_{j=r}^s a_j b_j (n-j-1) \alpha^2 \int_M F(\nu) (H^F H_{j+1}^F - H_{j+2}^F) dv_g \\
&\quad - \sum_{j=r}^s a_j b_j (j+1) \alpha^2 \int_M F(\nu) H^F H_{j+1}^F dv_g \\
&\quad + \sum_{j=r}^s 2a_j b_j (j+1) \alpha \beta \int_M F(\nu) H_{j+1}^F dv_g \\
(12) \quad &\quad + \sum_{j=r}^s a_j b_j (j+1) \beta^2 \int_M H_{j+1}^F \langle X, \nu \rangle dv_g.
\end{aligned}$$

Using the anisotropic Hsiung-Minkowski formulas again, we have

$$\sum_{j=r}^s 2a_j b_j (j+1) \beta^2 \int_M H_{j+1}^F \langle X, \nu \rangle dv_g = - \sum_{j=r}^s 2a_j b_j (j+1) \beta^2 \int_M F(\nu) H_j^F dv_g,$$

and therefore, (12) becomes

$$\begin{aligned}
\mathcal{J}_{r,s,F}''(0)[f] &= - \sum_{j=r}^s a_j b_j (n-j-1) \alpha^2 \int_M F(\nu) (H^F H_{j+1}^F - H_{j+2}^F) dv_g \\
(13) \quad &\quad - \sum_{j=r}^s a_j b_j (j+1) \int_M F(\nu) (H^F H_{j+1}^F \alpha^2 - 2H_{j+1}^F \alpha \beta + H_j^F \beta^2) dv_g
\end{aligned}$$

Now, at a point  $x$  on  $M$ , we consider the following second order polynomial

$$P_{j,F,x}(z) = F(\nu) (H^F H_{j+1}^F z^2 - 2H_{j+1}^F \beta z + H_j^F \beta^2).$$

The discriminant of  $P_{j,F,x}$  is

$$\Delta = 4\beta^2 F(\nu)^2 ((H_{j+1}^F)^2 - H^F H_{j+1}^F H_j^F) = 4\beta^2 F(\nu)^2 H_{j+1}^F (H_{j+1}^F - H^F H_j^F).$$

Since, by assumption,  $H_{s+1} > 0$ , by Lemma 2.2, we have that  $H_{j+1}^F > 0$  and  $H_{j+1}^F - H^F H_j^F \geq 0$ . Hence,  $\Delta$  is nonnegative and since the term of degree 2 is  $F(\nu) H^F H_{j+1}^F > 0$ , then  $P_{j,F,x}(z) \geq 0$  for any  $z \in \mathbb{R}$ . In particular, for  $z = \alpha$ , we obtain

$$H^F H_{j+1}^F \alpha^2 - 2H_{j+1}^F \alpha \beta + H_j^F \beta^2 \geq 0.$$

Reporting this in (13), we get

$$\mathcal{J}_{r,s,F}''(0)[f] \leq - \sum_{j=r}^s a_j b_j (n-j-1) \alpha^2 \int_M F(\nu) (H^F H_{j+1}^F - H_{j+2}^F) dv_g.$$

Finally, since  $F(\nu) > 0$  and  $H^F H_{j+1}^F - H_{j+2}^F \geq 0$  by Lemma 2.2, we get that

$$\mathcal{J}_{r,s,F}''(0)[f] \leq 0.$$

Since, by the  $(r, s, F)$ -stability assumption, we have  $\mathcal{J}_{r,s,F}''(0)[f] \geq 0$ , we deduce that

$$\mathcal{J}_{r,s,F}''(0)[f] = 0.$$

This means that each term in the right-hand side of (13) vanishes. In particular, we have

$$\sum_{j=r}^s a_j b_j (n-j-1) \alpha^2 \int_M F(\nu) (H^F H_{j+1}^F - H_{j+2}^F) dv_g = 0,$$

and so for each  $j \in \{r, \dots, s\}$ ,

$$\int_M F(\nu) (H^F H_{j+1}^F - H_{j+2}^F) dv_g = 0.$$

Since  $F(\nu) > 0$  and  $H^F H_{j+1}^F - H_{j+2}^F \geq 0$ , we get that at any point  $x$  of  $M$ ,

$$H^F H_{j+1}^F - H_{j+2}^F = 0.$$

Thus, by Lemma 2.2, we conclude that  $X(M)$  is the Wulff shape, up to translations and homotheties. This concludes the proof of Theorem 1.1.  $\square$

## REFERENCES

- [1] H. Alencar, M. do Carmo & A.G. Colares, *Stable hypersurfaces with constant scalar curvature*, Math. Z. **213** (1993), 117-131.
- [2] H. Alencar, M.P. Do Carmo & H. Rosenberg, *On the first eigenvalue of Linearized operator of the  $r$ -th mean curvature of a hypersurface*, Ann. Glob. Anal. Geom., **11** (1993), 387-395.
- [3] J.L.M. Barbosa & M. do Carmo, *Stability of hypersurfaces with constant mean curvature*, Math. Z. **185** (1984), 339-353.
- [4] J.L.M. Barbosa, M.P. Do Carmo & J. Eschenburg, *Stability of hypersurfaces with constant mean curvature in Riemannian manifolds*, Math. Z. **197** (1988), 123-138.
- [5] J.L.M. Barbosa & A.G. Colares, *Stability of hypersurfaces with constant  $r$ -mean curvature*, Ann. Glob. Anal. Geom. **15** (1997), 277-297.
- [6] A.G. Colares & J.F. da Silva, *Stable hypersurfaces as minima of the integral of an anisotropic mean curvature preserving a linear combination of area and volume*, Math. Z. **275** (2013), 595-623.
- [7] H. Chen & X. Wang, *stability and eigenvalue estimates of linear Weingarten hypersurfaces in a sphere*, J. Math. Anal. Appl. **397** (2) (2013), 653-670.
- [8] J.F. Grosjean, *Extrinsic upper bounds for the first eigenvalue of elliptic operators*, Hokkaido Math. J. **33** (2004), no. 2, 219-239.
- [9] H.F. de Lima & M.A.L. Velásquez, *A new characterization of  $r$ -stable hypersurfaces in space forms*, Arch. Math. (Brno) **47** (2011), 119-131.
- [10] Y. He & H. Li, *Integral formula of Minkowski type and new characterization of the Wulff Shape*, Acta Math. Sin. **24** (2008), no. 3, 697-704.
- [11] Y. He & H. Li, *A new variational characterization of the Wulff shape*, Diff. Geom. Appl., **26** (2008), no. 4, 377-390.
- [12] Y. He & H. Li *Stability of area-preserving variations in space forms*, Ann. Glob. Anal. Geom. **34** (2008), 55-68.
- [13] Y. He & H. Li, *Stability of hypersurfaces with constant  $(r+1)$ -th anisotropic mean curvature*, Illinois J. Math. **52** (4) (2008), 1301-1314.
- [14] Y. He, H. Li, H. Ma & J. Ge, *Compact embedded hypersurfaces with constant higher order anisotropic mean curvature*, Indiana Univ. Math. J. **58** (20058), 853-868.
- [15] C.C. Hsiung, *Some integral formulae for closed hypersurfaces*, Math. Scand **2** (1954), 286-294.
- [16] M. Koiso and B. Palmer, *Geometry and stability of surfaces with constant anisotropic mean curvature*, Indiana Univ. Math. J. **54** (2005), 1817-1852.
- [17] L. Onat, *Some characterizations of the Wulff shape*, C. R. Math. Acad. Sci. Paris **348** (2010), no. 17-18, 9971000.
- [18] B. Palmer, *Stability of the Wulff shape*, Proc. Amer. Math. Soc. **126** (1998), no. 2, 3661-3667.
- [19] H. Rosenberg, *Hypersurfaces of constant curvature in space forms*, Bull. Sc. Math. **117** (1993), 221-239.
- [20] J.F. da Siva, H.F. de Lima & M.A.L. Velásquez, *The stability of hypersurfaces revisited*, Monatsh. Math (in press) DOI 10.1007/s00605-015-0776-x.

- [21] M.A.L. Velásquez, A.F. de Sousa & H.F. de Lima, *On the stability of hypersurfaces in space forms*, J. Math. Anal. Appl. **406** (2013), 134-146.

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